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UNIT

1 C1 : Mean Value Theorem

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Previous Year Question
(2024 Pattern)

2. Rolle's Theorem

Q. Verify Rolle's theorem for $f(x)=x^2$, $x \in (-1,1)$ [Nov24]

3. Lagrange's Mean Value Theorem (LMVT)

Q. Prove for $0<a<b$: $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ [Nov24]

Q. Verify LMVT for $f(x) = \log x$ on $[1,e]$ [Apr25]

Q. Using LMVT show: $\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}b - \sin^{-1}a < \frac{b-a}{\sqrt{1-b^2}}$ [Apr25]

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◀ Mean Value Theorem ▶

- The Mean Value Theorem (MVT) helps to find a point where the slope of the tangent equals the average rate of change of the function.

- In simple words, it connects average change with instantaneous change (derivative).

- It is an important concept in calculus, used in many applications like error analysis, motion, and curve behavior.

There are four main forms of Mean Value Theorem:

1. Rolle's Theorem
2. Lagrange's Mean Value Theorem (LMVT)
3. Cauchy's Mean Value Theorem (CMVT)
4. Taylor's Theorem

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◀ Rolle's Theorem ▶

- Rolle's theorem tells us that if a function starts and ends at the same height (same y-value), then the slope of the tangent (derivative) becomes zero at least once in between.

- It is a special case of the Mean Value Theorem.

Conditions of Rolle's Theorem:

Let $f(x)$ be a real function defined on interval $[a, b]$. Then Rolle's Theorem applies only if:

1. $f(x)$ is continuous on the closed interval $[a, b]$. → means the graph has no breaks or jumps.
2. $f(x)$ is differentiable on the open interval (a, b). → means the curve is smooth (no sharp corners).
3. $f(a) = f(b)$. → function starts and ends at the same y-value. If all three conditions are satisfied, then there exists at least one point $c \in (a, b)$ such that

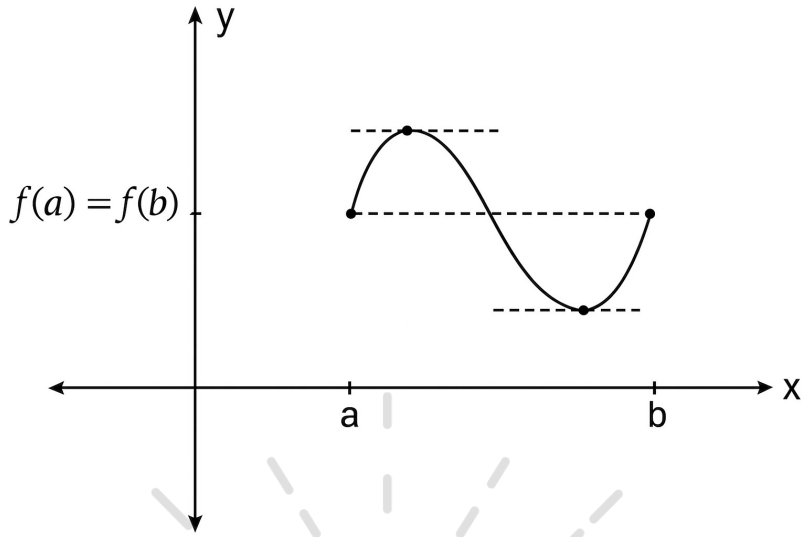
$$f'(c) = 0$$

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Basic Idea

- If a function is smooth (continuous + differentiable) and has the same y-value at the endpoints, then somewhere in between, the tangent line will be horizontal (slope = 0).

- Graphically, this means: → The tangent line is parallel to the X-axis at that point.



Formal Statement

If $f(x)$ is:

- continuous on $[a, b]$,
- differentiable on (a, b),
- and $f(a) = f(b)$, then there exists at least one point $c \in (a, b)$ such that

$$f'(c) = 0$$

Example: Let $f(x) = x^2 - 4x + 3$ on $[1,3]$

Step 1:

$$f(1) = 1 - 4 + 3 = 0$$

$$f(3) = 9 - 12 + 3 = 0$$

$$\checkmark f(a) = f(b)$$

Step 2:

$$f'(x) = 2x - 4$$

Set $f'(x) = 0$:

$$2x - 4 = 0 \Rightarrow x = 2$$

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✓$x = 2$ lies in (1,3)

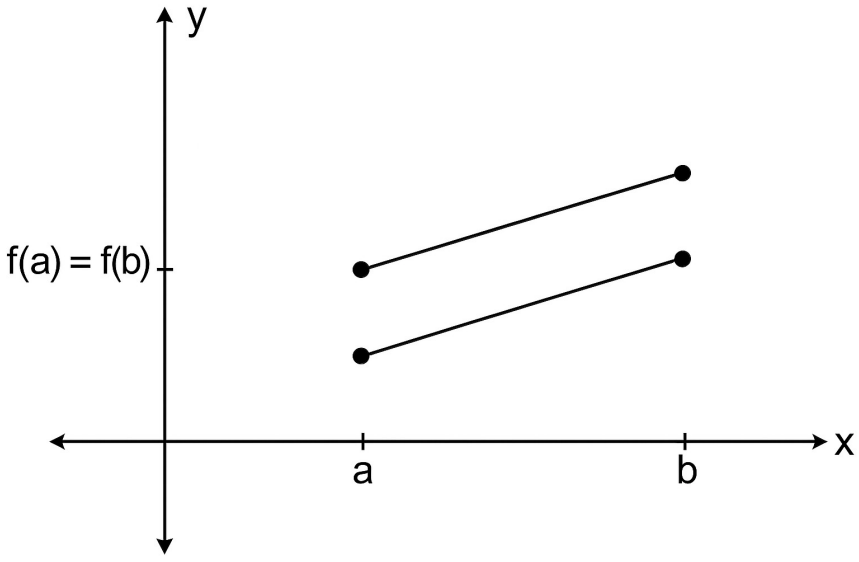
Hence, Rolle's theorem is verified

Exam Tips

- Always check three conditions first.
- Don't forget to verify $f(a) = f(b)$.
- For proof or verification type questions, show step-by-step:
- Verify continuity and differentiability
- Check $f(a) = f(b)$
- Find c such that $f'(c) = 0$

Understanding Rolle's Theorem

- If a function is not continuous on $[a, b]$, it can have breaks or jumps in its graph. Because of this, there might not exist any point where the tangent to the curve is horizontal (parallel to X-axis).
- So, for Rolle's Theorem to hold, the curve must be unbroken - i.e. continuous throughout $[a, b]$.



Ex 6)

◀ Lagrange's Mean Value Theorem (LMVT) ▶

- Lagrange's Mean Value Theorem (LMVT) connects the average rate of change of a function with its instantaneous rate of change (derivative).
- It states that for a smooth curve between two points, there exists at least one point where the tangent to the curve is parallel to the secant line joining those points.

Average Rate of Change = Instantaneous Rate of Change at some point c

Geometrically:

- LMVT shows a relationship between the slope of the secant line and the slope of the tangent line.
- The tangent at some point c in (a, b) is parallel to the secant joining $(a, f(a))$ and $(b, f(b))$.

Note: Rolle's Theorem is a special case of LMVT where $f(a) = f(b)$.

Conditions for LMVT:

LMVT holds true if the following two conditions are satisfied:

1. $f(x)$ is continuous on the closed interval $[a, b]$. → No breaks, jumps, or gaps.
2. $f(x)$ is differentiable on the open interval (a, b). → The curve is smooth (no sharp corners).

If these two conditions are satisfied, then:

There exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

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Formal Statement

If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists at least one value $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrical Meaning:

- The term $\frac{f(b)-f(a)}{b-a}$ represents the slope of the secant joining the end points of the curve.
- The derivative $f'(c)$ represents the slope of the tangent at some point c in between.
- LMVT guarantees that there is at least one such point c where both slopes are equal.

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◀ Solved Example on LMVT ▶

Ex 1) Suppose $f(x) = x^2$ is defined on the interval $[-1,2]$. Find the point on f where the tangent line is parallel to the secant line.

Soln:

Step 1: Verify two conditions

1&2. The given function $f(x) = x^2$ is a polynomial function,

so it is continuous everywhere and differentiable everywhere.

$\therefore f(x)$ is continuous on $[-1,2]$ and differentiable on $(-1,2)$

Step 2: Find end points

We have, $f(x) = x^2$

Put $= -1$:

$$f(-1) = (-1)^2 = 1$$

Put $= 2$:

$$f(2) = (2)^2 = 4$$

Now, the slope of the secant line is given by:

$$\text{Slope} = \frac{f(b) - f(a)}{b - a}$$

$$\text{Slope} = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{4 - 1}{2 + 1} = \frac{3}{3} = 1$$

$$\therefore \frac{f(b) - f(a)}{b - a} = 1$$

Step 3: Find derivative

$$f(x) = x^2$$

$$f'(x) = 2x$$

At $x = c$,

$$f'(c) = 2c$$

Step 4: Apply LMVT

According to LMVT,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$2c = 1$$

$$\therefore c = \frac{1}{2}$$

Step 5: Find the corresponding point

$$f(x) = x^2$$

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\therefore \text{Point is } \left(\frac{1}{2}, \frac{1}{4}\right)$$

Final Answer:

$$\boxed{\left(\frac{1}{2}, \frac{1}{4}\right)}$$

is the point on the curve where the tangent is parallel to the secant line.

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Ex 2) Verify LMVT for $f(x) = x(x-2)$ on $[1,3]$

Soln:

Step 1 : Verify two conditions

1&2. Since $f(x)$ is a polynomial function, it is continuous and differentiable everywhere.

$\therefore f(x)$ is continuous on $[1,3]$ and differentiable on $(1,3)$

Hence, by LMVT, there exists at least one value $c \in (1,3)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\therefore f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

Step 2 : Find $f'(x)$ We have,

$$f(x) = x(x-2)$$

$$\Rightarrow f(x) = x^2 - 2x$$

Differentiate w.r.t. :

$$f'(x) = 2x - 2$$

Put $= c$:

$$f'(c) = 2c - 2$$

Also, find $f(3)$ and (1) :

$$f(3) = 3(3-2) = 3$$

$$f(1) = 1(1-2) = -1$$

Step 3 : From Equation (1)

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$2c - 2 = \frac{3 - (-1)}{2}$$

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$$2c - 2 = \frac{4}{2}$$

$$2c - 2 = 2$$

$$2c\& = 4$$

$$\therefore c = 2$$

Since $c = 2 \in (1,3)$, LMVT is satisfied.

Step 4 : Find the corresponding y-coordinate

$$f(x) = x(x - 2)$$

$$f(2) = 2(2 - 2) = 0$$

Conclusion:

$$\boxed{x = 2, y = 0}$$

Hence, at the point $(2,0)$, the tangent line is parallel to the secant line and Lagrange's Mean Value Theorem is verified.

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Ex 3) Using Lagrange's Mean Value Theorem (LMVT), find a point on the curve

$$y = \sqrt{x-2}$$

defined on [2,3] where the tangent is parallel to the chord joining the end points on the curve.

Soln:

Step 1 : Verify LMVT conditions

Given,

$$y = \sqrt{x-2}$$

1. The function is continuous on [2,3] :

Because $\sqrt{x-2}$ is defined for all $x \geq 2$,

so y is continuous on [2,3].

2. The function is differentiable on (2,3) :

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x-2}}$$

which exists for $x > 2$.

Hence, y is continuous on [2, 3] and differentiable on (2, 3).
Therefore, LMVT is applicable.

Step 2 : Apply LMVT

By LMVT, there exists at least one $c \in (2,3)$ such that

$$f'(c) = \frac{f(3)-f(2)}{3-2} \quad \ldots(1)$$

Step 3 : Substitute the function values

$$f(x) = \sqrt{x-2}$$

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$$f(3) = \sqrt{3-2} = 1, \quad f(2) = \sqrt{2-2} = 0$$

From (1),

$$f'(c) = \frac{1-0}{3-2} = 1$$

But,

$$f'(x) = \frac{1}{2\sqrt{x-2}}$$

So,

$$\frac{1}{2\sqrt{c-2}} = 1$$

$$2\sqrt{c-2} = 1$$

$$\sqrt{c-2} = \frac{1}{2}$$

$$c - 2 = \frac{1}{4}$$

$$c = \frac{9}{4} \in (2,3)$$

Step 4 : Find corresponding y-coordinate

$$y = f(x) = \sqrt{x-2}$$

$$f\left(\frac{9}{4}\right) = \sqrt{\frac{9}{4} - 2} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

Therefore, the required point is

$$\boxed{\left(\frac{9}{4}, \frac{1}{2}\right)}$$

At this point, the tangent is parallel to the chord joining $(2,0)$ and $(3,1)$.

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Ex 4) Verify LMVT for $f(x) = x^2$ in $[1,5]$

Soln:

Step 1 : Verify the conditions

(i) $f(x) = x^2$ is a polynomial function, hence it is continuous and differentiable everywhere.

$\Rightarrow f(x)$ is continuous on $[1,5]$

and differentiable on $(1,5)$

Therefore, by LMVT, there exists at least one $c \in (1,5)$ such that

$$f'(c) = \frac{f(5) - f(1)}{5 - 1}$$

Step 2 : Find $f'(x), f(5)$, and $f(1)$

$$f(x) = x^2$$

Differentiate:

$$f'(x) = 2x$$

$$f(5) = 25, \quad f(1) = 1$$

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Step 3 : Substitute in (1)

$$f'(c) = \frac{25 - 1}{5 - 1}$$

$$2c = 6$$

$$c = 3$$

Since $3 \in (1,5)$, LMVT holds.

Step 4 : Find the corresponding point

$$f(x) = x^2$$

$$f(3) = 3^2 = 9$$

Therefore, the point is

$$\boxed{(3,9)}$$

At this point, the tangent is parallel to the chord joining $(1,1)$ and $(5,25)$.

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◀ Cauchy's Mean Value Theorem (CMVT) ▶

- The Cauchy's Mean Value Theorem (CMVT) is an extension of Lagrange's Mean Value Theorem (LMVT).

- It deals with two functions, not just one, and shows how their derivatives are related.

It establishes a relationship between the rate of change of two functions $f(x)$ and $g(x)$.

$$\text{i.e., } \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

or equivalently,

$$f'(c) = kg'(c) \quad \text{where } k = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Basic Idea:

Suppose we have two functions $f(x)$ and $g(x)$ defined on the same interval $[a, b]$:

- $f(b) - f(a) = 6$
- $g(b) - g(a) = 3$

This means that the change in f is twice the change in g**.

So, there must exist at least one point c between a and b such that:

$$f'(c) = 2g'(c)$$

In general, we can write:

$$f'(c) = kg'(c)$$

where k is the constant ratio of their total changes.

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Geometrically:

- CMVT says that at some point c, the ratio of the slopes of the tangents to $f(x)$ and $g(x)$ is the same as the ratio of their overall changes between a and b.

Connection with LMVT

- If we take $g(x) = x$, then $g'(x) = 1$.
- Substituting in CMVT formula:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

which is the Lagrange's Mean Value Theorem
Hence, LMVT is a special case of CMVT.

Formal Statement (Version 1)

If $f(x)$ and $g(x)$ satisfy the following conditions:

5. $f(x)$ and $g(x)$ are continuous on $[a, b]$
6. $f(x)$ and $g(x)$ are differentiable on (a, b)
7. $g(a) \neq g(b)$

Then, there exists at least one value $c \in (a, b)$ such that:

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}}$$

or

$$f'(c) = \frac{f(b) - f(a)}{g(b) - g(a)} g'(c)$$

$[a, b]$

Understanding the Equation (Simple Analogy)

Think of two boats, A and B, racing over the same time interval [a,b]

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- $f(b) - f(a)$: total distance covered by boat A
- $g(b) - g(a)$: total distance covered by boat B

If the ratio $\frac{f(b)-f(a)}{g(b)-g(a)} = 2$, it means boat A travels twice as far as boat B in the same time. At some instant $= c$:

$$\frac{f'(c)}{g'(c)} = 2$$

which means the speed of boat A is twice that of boat B.

So, CMVT guarantees that there is at least one such moment c.

Why End Points Must Be Unequal (in Version 1)

- We require $g(a) \neq g(b)$ to avoid division by zero in $\frac{f(b)-f(a)}{g(b)-g(a)}$.
- This is why $g(x)$ cannot have equal end values in the first version.
- But this restriction is removed in Version 2 (next).

Formal Statement (Version 2)

If $f(x)$ and $g(x)$ are:

8. Continuous on $[a, b]$, and
9. Differentiable on (a, b), then there exists a value $c \in (a, b)$ such that:

$$\boxed{[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c)}$$

This version removes the need for $g(a) \neq g(b)$ because it doesn't involve division - only multiplication.

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Ex 5) Show that there exists a number $c \in (a, b)$ such that

$$2c[f(a) - f(b)] = f'(c)[a^2 - b^2]$$

when $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b).

Soln:

Let

$$f(x) = f(x) \quad \text{and} \quad g(x) = x^2$$

Both $f(x)$ and $g(x)$ are (i) Continuous on $[a, b]$ (ii) Differentiable on (a, b)

Hence, by the Cauchy Mean Value Theorem (CMVT), there exists a value $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Now,

$$g'(x) = 2x \Rightarrow g'(c) = 2c$$

Substitute this in the CMVT equation:

$$\frac{f'(c)}{2c} = \frac{f(b) - f(a)}{b^2 - a^2}$$

Multiply both sides by $2c(b^2 - a^2)$:

$$f'(c)[b^2 - a^2] = [f(b) - f(a)] \cdot 2c$$

Rearranging, we get:

$$2c[f(a) - f(b)] = f'(c)[a^2 - b^2]$$

Hence proved

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